Department of Physics
PRELIMINARY EXAMINATION 2018
Part II. Long Questions/Answers

Friday May 18th, 2018, 14-17h

Examiners: Prof. J. Cline, Prof. K. Dasgupta, Prof. H. Guo, Prof. G. Gervais (Chair), Prof. D. Hanna, and Prof. V. Kaspi.

INSTRUCTIONS

Answer **4** questions out of the choice of **8**. You must attempt one problem in each category, *i.e.* one in *classical mechanics/special relativity*, one in *quantum mechanics*, one in *electromagnetic theory* and one in *statistical mechanics/thermo.*

This is a **closed book** exam. Approved calculators may be used (non-programmable ones), though approximate numerical results are valid.

If you attempt more than four questions, you should clearly mark which ones should be graded.

DO NOT WRITE YOUR NAME; write **ONLY** you student ID on the exam booklet. Clearly indicate the question number next to each answer.

This exam has 9 pages, including this title page.

Statistical Mechanics/Thermodynamics

Statistical magnets

A material consists of n independent particles and is in a weak external magnetic field H. Each particle can have a magnetic moment $m\mu$ along the magnetic field, where $m = J, J-1, ..., -J+1, -J$, with J being an integer and μ is a constant. The system is at a temperature T.

(a) Find the partition function for this system.

(b) Calculate the average magnetization $< M >$ of the material.

(c) For large values of T find an asymptotic expression for $< M >$.

Statistical mechanics in the sky

Model a neutron star as being a sphere of radius R containing a degenerate Fermi gas of N neutrons; the neutron mass is m_n.

(a) Compute the Fermi energy E_F, assuming the neutrons are nonrelativistic.

(b) Show that the total kinetic energy of the star is $U = (3/5)NE_F$. Express it as a function of R.

(c) The gravitational potential energy is $U_{\text{grav}} = -(3/5)GM^2/R$ where $M = Nm_n$ is the mass of the star. By minimizing the total energy with respect to R, find the size of the star, in terms of M, m_n and fundamental constants.

Quantum Mechanics

Three spins in Hilbertland

Consider a system of three spin-$\frac{1}{2}$ particles. The Hilbert space is spanned by the basis vectors $|\pm z, \pm z, \pm z\rangle \equiv |\pm z\rangle_1 |\pm z\rangle_2 |\pm z\rangle_3$.

(a) Suppose $|s, m\rangle$ denotes the state of the three particles having total spin s and z-component of the total spin m, determine $|\frac{3}{2}, -\frac{1}{2}\rangle$ in the basis of $|\pm z, \pm z, \pm z\rangle$.

(b) Now, the three spin-$\frac{1}{2}$ particles interact with each other by pair-wise dipole interaction:

$$\hat{H} \;=\; \frac{2A}{\hbar^2}\left(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 \;+\; \hat{\mathbf{S}}_2 \cdot \hat{\mathbf{S}}_3 \;+\; \hat{\mathbf{S}}_3 \cdot \hat{\mathbf{S}}_1\right) \;,$$

write down the matrix representation of $\hat{H}$ in the basis of $|\pm z, \pm z, \pm z\rangle$.

Note the problem is less tedious by working out the pair interaction in terms of the raising/lowering operators. Then, by deriving matrix elements of one pair, you can probably guess the other matrix elements. The following expressions may be useful: $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$. $[\hat{S}_z, \hat{S}_\pm] = \pm\hbar\hat{S}_\pm$. $\hat{S}^2|s,m\rangle = s(s+1)\hbar^2|s,m\rangle$, $\hat{S}_z|s,m\rangle = m\hbar|s,m\rangle$.

A fine time-dependent photon factory

A hypothetical interaction can be modeled as an oscillating function $a(t) = a_0 \cos \omega t$ coupled to electric and magnetic fields through the interaction Hamiltonian

$$H_{\text{int}} = a(t) \int d^3x \boldsymbol{E} \cdot \boldsymbol{B}$$

In an external magnetic field, this can produce photons. We denote a state with a photon of wave vector $\boldsymbol{k}$ and polarization vector $\boldsymbol{\epsilon}_k^{(i)}$ (where $i = 1, 2$ label the two possible polarizations transverse to $\boldsymbol{k}$) by $|\boldsymbol{k}, i\rangle$. The photon frequency is $\omega_k = ck$, with c the speed of light. It can be shown that the transition matrix element of H_{int} between the initial vacuum and the final one-photon state $|\boldsymbol{k}, j\rangle$ is

$$H_{fi} = i\omega_k a_0 c \sqrt{\frac{\hbar}{2kV}}\, \boldsymbol{\epsilon}_k^{(j)} \cdot \boldsymbol{B}(k)\, e^{i\omega_k t} \cos \omega t$$

where $\boldsymbol{B}(k)$ denotes the Fourier tranform of $\boldsymbol{B}(x)$, and V is a fictitious volume containing the experiment, used to define the density of photon states.

(a) Using time-dependent perturbation theory, show that the amplitude for creating the photon is given by

$$D_j \left(\frac{e^{i(\omega-\omega_k)t} - 1}{i(\omega - \omega_k)} + \frac{e^{-i(\omega+\omega_k)t} - 1}{-i(\omega + \omega_k)} \right)$$

where D_j is a constant independent of time (but it depends on $\boldsymbol{k}$ and the photon polarization). Identify D_j using the result of part (a). Assuming that $\omega > 0$ and $\omega_k > 0$, which term will give the dominant contribution at times much greater than $1/\omega$ or $1/\omega_k$? Square this dominant term to find the time dependence of the transition probability at late times.

(b) Given that the density of states for photons in a box of volume V is

$$dN = 2V \frac{d^3k}{(2\pi)^3},$$

integrate the transition probability from (a) over the final photon states and show that it grows linearly with time, giving a constant rate, in accordance with Fermi's Golden rule. Hint: $\int_{-\infty}^{\infty} dx \sin^2 x/x^2 = \pi$. Find the rate, in terms of a remaining integral over the photon directions.

Electromagnetic Theory

Conducting hemisphere trying to split up

An electrically neutral, conducting, spherical shell of radius R is placed in a uniform electric field $\mathbf{E}_o$. Assume $\mathbf{E}_o$ along the z-axis.

(a) Write down the electrostatic potential ϕ at radial coordinate $r = R$ and $r \to \infty$ (recall $\mathbf{E} = -\nabla\phi$).

(b) Find ϕ as a function of the radial coordinate r and the polar angle θ of spherical coordinates.

(c) Find the surface charge density σ on the conducting shell.

(d) Now, suppose the sphere is cut into two hemispheres by a plane perpendicular to $\mathbf{E}_o$. Find the force that would be needed to prevent the two hemispheres from moving apart.

Coulomb is summing up in 1D

A simple 1-dimensional model for an ionic crystal (such as NaCl) consists of an array of N point charges in a straight line, alternately $+e$ and $-e$ and each at a distance a from its nearest neighbours. If N is very large, find the potential energy of a charge in the middle of the row and of one at the end of the row in the form $\alpha e^2/(4\pi\epsilon_0 a)$. This sets upper and lower bounds to the Coulomb energy of the whole array. Here, α is known as the *Madelung constant* for such systems.

Hint: The following mathematical relation may be useful for writing α in a closed form: $ln(1+x) = x - x^2/2 + x^3/3 - ...$ for $-1 < x \leq 1$.

Classical Mechanics

The angle of a relativistic decay event

The Λ^0 particle (mass = 1115 MeV/c^2) decays into a proton (mass = 938 MeV/c^2) and a π^- particle (mass = 140 MeV/c^2). Suppose the Λ^0 is travelling along the x axis in the laboratory frame, S, and has momentum of 20 GeV/c (20000 MeV/c) in this frame.

(a) What is the energy of the π^- in S', the rest frame of the Λ^0?

(b) If the π^- has 3-momentum $\mathbf{p'} = (p'/\sqrt{2}, p'/\sqrt{2}, 0)$ in the Λ^0 rest frame, what will be the angle of its track with respect to the x axis in S? (Assume the x and x' axes are collinear.)

(c) What will be the angle of the proton's track in S?

Exploding classical mechanics

Consider a binary star system initially consisting of one star of mass M_1 and one of mass M_2, separated by distance a, in circular orbits about their common centre of mass. Their orbital speeds are v_1 and v_2, respectively.

Now suppose star 1 explodes in a supernova, and assume the explosion is symmetric so that there is no change in v_1, and initially after the explosion, the stellar separation has not changed. Assume also that star 1 leaves behind a compact stellar remnant of mass M_R, and the remaining mass, ΔM, is blown off into space.

Show that the binary system becomes unbound if more than one-half of the total initial mass $(M_1 + M_2)$ is lost from the system, *i.e.* if

$$\Delta M > (M_1 + M_2)/2. \tag{1}$$

(*Hint: consider initial and final energies, and use the Virial Theorem.)*